

On Convex Geometric Graphs with no $k + 1$ Pairwise Disjoint Edges

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Abstract

A well-known result of Kupitz from 1982 asserts that the maximal number of edges in a convex geometric graph (CGG) on n vertices that does not contain $k + 1$ pairwise disjoint edges is kn (provided $n > 2k$). For $k = 1$ and $k = n/2 - 1$, the extremal examples are completely characterized. For all other values of k , the structure of the extremal examples is far from known: their total number is unknown, and only a few classes of examples were presented, that are almost symmetric, consisting roughly of the kn “longest possible” edges of $CK(n)$, the complete CGG of order n .

In order to understand further the structure of the extremal examples, we present a class of extremal examples that lie at the other end of the spectrum. Namely, we break the symmetry by requiring that, in addition, the graph admit an independent set that consists of q consecutive vertices on the boundary of the convex hull. We show that such graphs exist as long as $q \leq n - 2k$ and that this value of q is optimal.

We generalize our discussion to the following question: what is the maximal possible number $f(n, k, q)$ of edges in a CGG on n vertices that does not contain $k + 1$ pairwise disjoint edges, and, in addition, admits an independent set that consists of q consecutive vertices on the boundary of the convex hull? We provide a complete answer to this question, determining $f(n, k, q)$ for all relevant values of n, k and q .

1 Introduction

A *geometric graph* (GG) is a graph whose vertices are points in general position in the plane, and whose edges are segments connecting pairs of vertices. If the vertices are in convex position (i.e., no vertex lies in the convex hull of the remaining vertices), the graph is called a *convex geometric graph* (CGG).

One of the oldest Turán-type questions for geometric graphs, mentioned already by Erdős [9] in 1946, is: what is the maximal number of edges in a geometric graph on n vertices that does not contain $k + 1$ *pairwise disjoint* edges? This question, along with the “dual” question concerning geometric graphs that do not contain $k + 1$ *pairwise crossing* edges, became central research topics in geometric combinatorics, with dozens of papers obtaining upper and lower bounds and resolving special cases (see, e.g., [2, 3, 6, 7, 8, 10, 11]).

For the sake of brevity, we call a GG that does not contain $k + 1$ pairwise disjoint edges an I_{k+1} -free graph, and a GG that does not contain $k + 1$ pairwise crossing edges an X_{k+1} -free graph.

For general GGs, the questions of determining the maximal possible number of edges in an I_{k+1} -free graph and in an X_{k+1} -free graph are still widely open. For I_{k+1} -free GGs, the best

currently known upper bound is $2^8 k^2 n$, obtained by Felsner [10], and the best lower bound is $\frac{3}{2}(k-1)n - 2k^2$, obtained by Tóth and Valtr [17]. For X_{k+1} -free CGGs, the best known upper bound is $2^{ck^6} n \log n$, obtained by Fox, Pach, and Suk [11]. For both questions, it is conjectured that the correct answer is of order $\Theta(kn)$ (see [16]).

Much more is known in the convex case. For X_{k+1} -free CGGs, Capovileas and Pach [6] showed in 1992 that the maximal possible number of edges is $2kn - \binom{2k+1}{2}$, for any $n \geq 2k+1$. It was also shown that any X_{k+1} -free CGG that is maximal with respect to inclusion (i.e., addition of any edge gives rise to $k+1$ pairwise crossing edges) contains exactly $2kn - \binom{2k+1}{2}$ edges [8], and the exact number of extremal examples was shown to be equal to the determinant of a certain matrix of Catalan numbers [13]. Strong upper bounds were obtained also for related questions, such as determining the maximal possible number of edges in a CGGs that does not contain a k -grid (i.e., two families of k edges each such that each edge in the first family crosses each edge in the other family), see [1, 5].

For I_{k+1} -free CGGs, Kupitz [15] showed in 1982 that the maximal possible number of edges is kn (for $n \geq 2k+1$) or $\binom{n}{2}$ (for $0 \leq n \leq 2k+1$). In the cases $k=1$ (for all n) and $k=n/2-1$ (for even n), the exact characterization of all *maximal* I_{k+1} -free CGGs (i.e., all I_{k+1} -free CGGs with exactly kn edges) is known. For $k=1$, Woodall [18] showed that the maximal CGGs are all self-intersecting circuits of odd order with inward pointing leaves. This characterization holds also for general (i.e., not necessarily convex) geometric graphs, and is related to Conway's *thrackle conjecture* (see [5], p. 306). For $k=n/2-1$, a characterization was given recently in [14]. The extremal graphs turn out to be complements of trees of a special structure called *caterpillar trees* (see [12]).

As for characterization of the maximal I_{k+1} -free CGGs for $1 < k < n/2-1$, the knowledge is very scarce. Unlike the case of maximal X_{k+1} -free CGGs, the number of maximal I_{k+1} -free CGGs is not known, and only a few examples of such CGGs were given (see, e.g., [10]). Moreover, all known examples are of a very specific kind, containing roughly the kn edges of maximal order in $CK(n)$, the complete CGG on n vertices (see Section 2 for the formal definition of "order"). If $CK(n)$ is represented by a regular polygon P of order n with all its diagonals, such graphs are almost fully symmetric about the center of the polygon, and they do not contain an independent set of more than $\lceil n/2 \rceil - k$ consecutive vertices on the boundary of P .

In this paper, we seek to broaden our acquaintance with the family of maximal I_{k+1} -free CGGs. To this end, we consider examples that are as far as possible from being centrally symmetric. Formally, we ask how large can be q , such that there exists a maximal I_{k+1} -free graph that contains an independent set of q consecutive vertices on the boundary of P . Such a restriction forces the graph to be far from symmetric, as demonstrated in Figure 1. We show by an explicit class of examples that q can be as large as $n-2k$ (i.e., twice as large as the value of q for the previously known examples), and prove that this value is maximal.

We obtain this result as a special case of a more general result concerning I_{k+1} -free graphs with a large independent set of consecutive boundary vertices.

Definition 1. Let G be a CGG. A set $A \subset V(G)$ is a free boundary arc of order q if:

- A consists of q consecutive vertices on the boundary of $\text{conv}(V(G))$,
- A is independent, i.e., no edge of G connects two vertices of A .

We often say that G *avoids* A , when A is an independent set of vertices of G .

The general question we address can be formulated as follows:

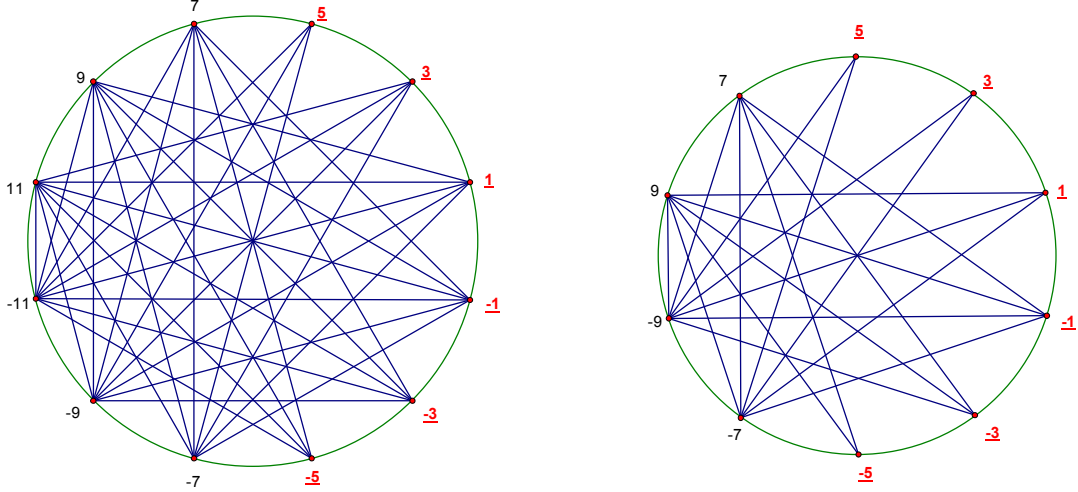


Figure 1: Two maximal CGGs that admit a free boundary arc of order $n - 2k$. On the left, $n = 12$ and $k = 3$, and on the right, $n = 10$ and $k = 2$. The avoided vertices are colored red and underlined.

Question 2. *What is the maximal number of edges in an I_{k+1} -free CGG on n vertices that admits a free boundary arc of order q ?*

We provide a complete answer in the following theorem:

Theorem 3. *Let n, k , and q be integers such that $1 \leq k \leq \lfloor n/2 \rfloor - 1$ and $1 \leq q \leq n - 1$. Denote by $f(n, k, q)$ the maximal number of edges in an I_{k+1} -free CGG on n vertices that admits a free boundary arc of order q . Then:*

1. *If $q \leq n - 2k$, then $f(n, k, q) = kn$. In particular, there exists a maximal I_{k+1} -free CGG that admits a free boundary arc of order q .*
2. *If $q = n - 2k + \ell$ for $0 < \ell < k$, then $f(n, k, q) = kn - \binom{\ell+1}{2}$.*
3. *If $q \geq n - k$, then $f(n, k, q) = \binom{n}{2} - \binom{q}{2}$.*

The proof that the stated values are upper bounds for $f(n, k, q)$ is quite straightforward. The proof of the equalities is somewhat more complex, and involves a construction of an asymmetric maximal I_{k+1} -free CGG. To illustrate the construction, two examples of CGGs with kn edges that admit a free boundary arc of order $n - 2k$ are given in Figure 1.

This paper is organized as follows: In Section 2 we introduce a few definitions and notations, and prove the upper bound of Theorem 3. In Section 3 we present a family of maximal I_{k+1} -free graphs $G_{n,k}$ that avoid $q = n - 2k$ consecutive vertices, thus proving Theorem 3(1). In Section 4 we modify the construction to produce I_{k+1} -free graphs $G_{n,k,\ell}$ that avoid $q = n - 2k + \ell$ consecutive vertices, and use this construction to complete the proof of Theorem 3.

2 Preliminaries

We start with a few definitions and notations, which are a bit unusual but will be very convenient for the proofs in the sequel.

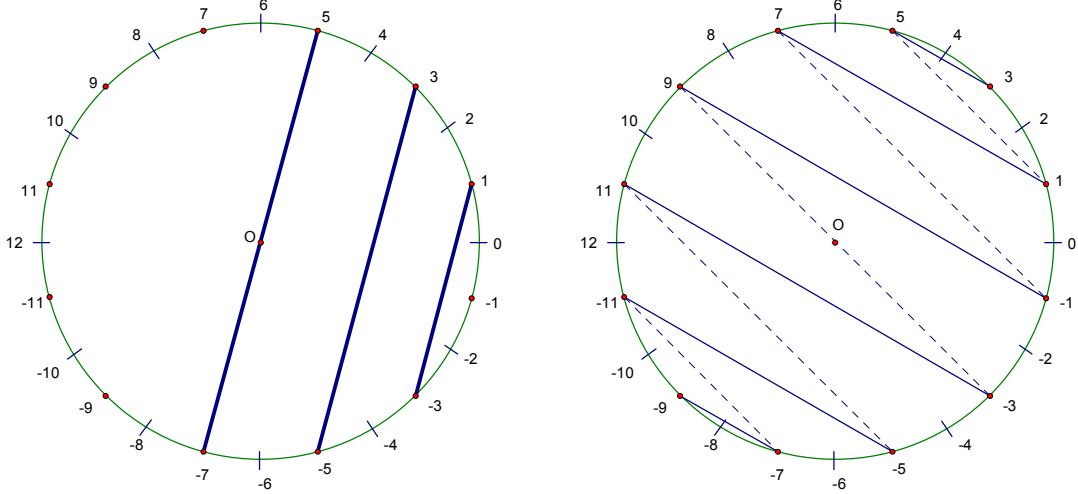


Figure 2: Representation of a CGG G on $n = 12$ vertices using a regular polygon P on 24 vertices, in the case of even $|A|$. The vertices of G are the odd-labelled vertices of P . On the right, all possible edges of direction 4 are drawn as ordinary lines and all possible edges of direction 3 are drawn as dotted lines. On the left, the set of bold edges is $B_{-1,5}$ (assuming $k = 3$).

2.1 Definitions and Notations

Let n be a positive integer, $n \geq 4$. In order to deal with a convex geometric graph $G = (V(G), E(G))$ of order n with a free arc A , we consider a regular polygon P of order $2n$, whose vertices lie on a circle with center O and are numbered cyclically from $-n + 1$ to n , as shown in Figure 2. We shall always insist that the free arc A is symmetric with respect to the horizontal axis through O and lie on the right. Thus, if $|A|$ is even, then we let $V(G)$ be the set of odd-labelled vertices of P , and if $|A|$ is odd, we let $V(G)$ be the set of even-labelled vertices of P . The segment connecting the endpoints of A will always be vertical.

Given the set A avoided by G , we say that an edge e is *allowed* if at least one of its endpoints lies in A^c (i.e., $V(G) \setminus A$). Otherwise, e is called *forbidden*. We also call the vertices of A *avoided* or *forbidden* vertices, and the vertices of A^c *allowed* vertices.

We define the *order* of an edge $[i, j] \in E(G)$ to be $ord([i, j]) = \frac{1}{2} \min(|i - j|, 2n - |i - j|)$. Note that each edge $e \in E(G)$ divides the boundary of P into two open arcs. If $ord(e) = r$, then these two arcs contain $r - 1$ and $n - r - 1$ vertices of G , respectively.

We say that an edge $e = [i, j] \in E(G)$ *emanates from* its left vertex, according to the labelling in Figure 2. Formally, e emanates from i if $|i| > |j|$. In the case $i = -j$, we say that the edge e emanates from the positive-labelled vertex.

For each vertex v of P , we consider the set of edges on $V(G)$ that are perpendicular to the radius \overrightarrow{Ov} . The edges of this set are parallel. If n is odd, then their number is $(n - 1)/2$. If n is even, then their number is $n/2$ for $v \in \text{vert}(P) \setminus V(G)$ and $n/2 - 1$ for $v \in V(G)$. We say that these are the edges *in direction* v . Note that in this notation, the vertices labelled i and $i - n$ (for any $1 \leq i \leq n$) correspond to the same direction. See the right part of Figure 2 for an illustration of the directions in a CGG on 12 vertices.

A pair of parallel edges on $V(G)$ is called *consecutive* if each vertex of one edge is adjacent on the boundary of $V(G)$ to a vertex of the other edge. Throughout the paper, we make use of

sequences of consecutive parallel edges (i.e., sequences of the form $\langle e_1, e_2, \dots, e_\ell \rangle$, where e_i, e_{i+1} are consecutive for each $1 \leq i \leq \ell - 1$). Of course, all edges in such a sequence belong to the same direction. For each such sequence S belonging to the direction i , the endpoints of the extremal edges of S determine two disjoint open arcs on the boundary of P that contain all the remaining vertices of G . We denote the arc that contains the vertex i by $Arc_{S,i}$ and the other arc by $Arc'_{S,i}$. The following notation will be important in the sequel.

Notation 4. For a direction i , and for $0 \leq j \leq n - 2k$, let $B_{i,j}$ denote the set S of k consecutive edges on $V(G)$ in direction i , for which exactly j vertices of G lie in $Arc'_{S,i}$. (This implies: $|V(G) \cap Arc_{S,i}| = n - 2k - j$, and therefore restricts the parity of j : we have $j \equiv n \pmod{2}$ if and only if i is **not** a label of a vertex of G .)

For example, in the left part of Figure 2, the set of edges depicted in bold is $S = B_{-1,5}$, and the two corresponding arcs on the boundary of P are $Arc_{S,-1}$ (containing the vertex -1) and $Arc'_{S,-1}$ (containing the vertices $7, 9, 11, -11, -9$). We note that, by its definition, the set $B_{i,j}$ depends also on n and k . In order to keep the notation concise, we prefer to leave this dependence implicit, as the values of n, k can always be understood from the context.

2.2 An Upper Bound on the Number of Edges

2.2.1 Upper bound for $q \leq n - 2k$

It is clear that all the edges in each direction are pairwise disjoint. Hence, if a set of edges does not contain $k + 1$ pairwise disjoint edges, then it contains at most k edges in each direction. As the number of distinct directions is n , we get the following observation (that was already used by Kupitz [15]):

Observation 5. Let G be an I_{k+1} -free CGG of order n . Then $|E(G)| \leq kn$. Moreover, if $|E(G)| = kn$, then $E(G)$ contains exactly k edges in each direction.

This implies the upper bound of Theorem 3(1).

2.2.2 Upper bound for $n - 2k < q < n - k$

Assume now that, in addition, G avoids a set A of $q = n - 2k + \ell$ consecutive vertices, for $1 \leq \ell < k$. In this case, the maximal possible number of edges is reduced, since, in some of the directions, only fewer than k edges have at least one endpoint in A^c . For $0 < s \leq k$, we say that direction j *loses* s edges, if it contains exactly $k - s$ edges with at least one endpoint in A^c , and denote this situation by $Loss(j) = s$. (If the number of edges in direction j with at least one endpoint in A^c is greater than or equal to k , we say that $Loss(j) = 0$.) In the proof, we use the following fact.

Claim 6. Let $n \in \mathbb{N}$. Then

$$\lceil n/2 \rceil + 2 \sum_{i=1}^{n-1} \lceil i/2 \rceil = \binom{n+1}{2}.$$

The proof, by induction on n , is straightforward and is left to the reader.

Let the allowed set A^c be as described in the right part of Figure 3, where $\pm x$ are its rightmost vertices. (Note that x is a function of n, k , and ℓ .) We compute the number of allowed edges in each direction.

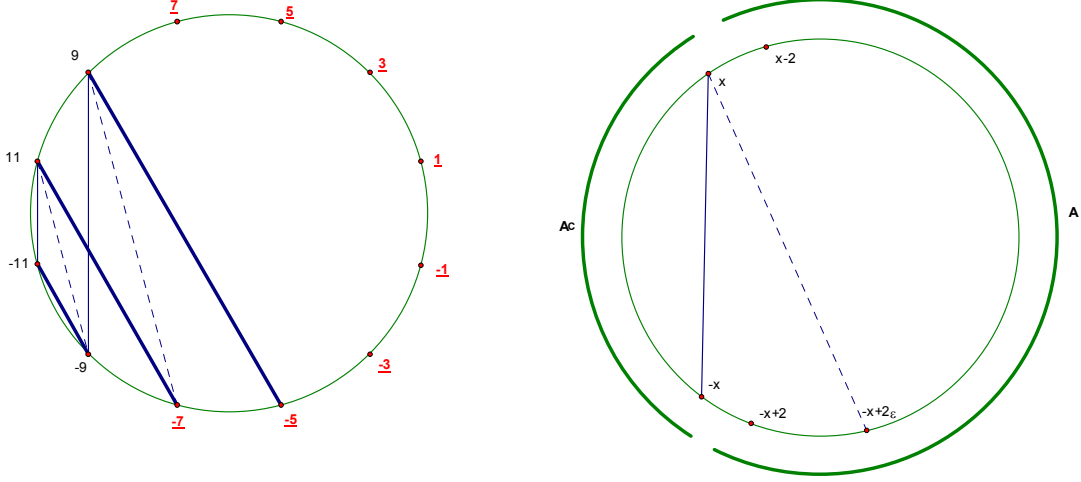


Figure 3: An illustration to the upper bound proof. The right drawing illustrates the proof for general parameters. The left drawing contains the special case $n = 12$, $k = 3$, and $\ell = 2$. The avoided vertices are colored red and underlined. The allowed edges in direction 0 are drawn as ordinary lines, the allowed edges in direction 1 are drawn as punctured lines, and the allowed edges in direction 2 are drawn as bold lines.

The endpoints of each edge in direction 0 are either both allowed or both forbidden. Hence, the number of allowed edges in direction 0 is $\lfloor (2k - \ell)/2 \rfloor = k - \lceil \ell/2 \rceil$, or in other words, $Loss(0) = \lceil \ell/2 \rceil$. (Note that if ℓ is odd, the leftmost vertex in A^c is idle.)

In direction ϵ , $0 < \epsilon \leq \ell$, the edge $e = [-x + 2\epsilon, x]$ is allowed, and so are all edges that are parallel to it and behind it (i.e., farther than e from ϵ). Thus, effectively, the allowed arc is extended by ϵ vertices on its bottom side, and hence, the number of allowed edges in this direction is $\lfloor (2k - \ell + \epsilon)/2 \rfloor$. Hence, $Loss(\epsilon) = \lceil (\ell - \epsilon)/2 \rceil$.

When ϵ reaches ℓ , the effective length of the allowed arc becomes $2k - \ell + \ell = 2k$, and thus, the loss becomes zero. By symmetry, the situation for direction $-\epsilon$ is the same as for direction ϵ .

Therefore, the sum of the losses is

$$\lceil \ell/2 \rceil + 2(\lceil (\ell - 1)/2 \rceil + \lceil (\ell - 2)/2 \rceil + \dots + \lceil 0/2 \rceil) = \ell(\ell + 1)/2,$$

where the equality holds by Claim 6.

The argument above is exemplified in the left drawing of Figure 3 in the special case $n = 12$, $k = 3$, $\ell = 2$. In direction 0 there are only 2 allowed edges, which corresponds to $Loss(0) = \lceil \ell/2 \rceil = 1$. The same holds for $\epsilon = 1$, and this corresponds to $Loss(1) = \lceil (\ell - \epsilon)/2 \rceil = \lceil (2 - 1)/2 \rceil = 1$. For $\epsilon = 2 = \ell$, there are already 3 allowed edges, which indeed corresponds to $Loss(2) = \lceil (\ell - 2)/2 \rceil = 0$.

2.2.3 Upper and lower bounds for $q \geq n - k$

Finally, we consider the case where G avoids a set A of $q \geq n - k$ consecutive vertices. In this case, each edge of G uses at least one of the vertices of A^c , and since $|A^c| \leq k$, this implies that G is I_{k+1} -free. Hence, the only restriction on G is the avoidance of A , which leads to the upper bound $\binom{n}{2} - \binom{q}{2}$ on the number of edges (as there are $\binom{q}{2}$ edges with both endpoints in A).

On the other hand, the upper bound is clearly attained by the graph G that contains all $\binom{n}{2} - \binom{q}{2}$ edges with at least one endpoint in A^c , since this graph is I_{k+1} -free and avoids A . This proves Theorem 3(3).

3 Maximal CGGs Avoiding $n - 2k$ Consecutive Vertices

In this section we consider I_{k+1} -free CGGs on n vertices that avoid a set A of $n - 2k$ consecutive vertices. By Observation 5, the number of edges in such a graph is at most kn . We show that this upper bound is attained by a sequence of graphs $G_{n,k}$, thus proving Theorem 3(1).

In Sections 3.1 and 3.2 we present the construction of $G_{n,k}$ for even values of n and prove that these CGGs satisfy the conditions of Theorem 3(1). In Section 3.3 we sketch the modifications in the construction of $G_{n,k}$ and in the proof of the theorem required in the case of odd n .

3.1 Definition of the Graphs $G_{n,k}$ for Even n

Let n be an even integer. Define $m = (n - 2k)/2$, and let the free set be $A = \{\pm 1, \pm 3, \dots, \pm(2m - 1)\}$.

Recall that by the definition of *direction* given in Section 2.1, directions i and $i - n$ (for any $1 \leq i \leq n$) are the same. Hence, in this section we restrict ourselves to the set of n directions that correspond to the half-circle consisting of the consecutive vertices $\{-m, -m + 1, \dots, 0, 1, \dots, m + 2k - 2, m + 2k - 1, m + 2k\}$ (where direction $m + 2k$ is equal to direction $-m$).

The CGG $G_{n,k}$ that avoids A is given by the following definition.

Definition 7. For an even integer n , and $k \leq \frac{n}{2} - 1$, we denote by $G_{n,k}$ the CGG on n vertices whose edge set consists of the following sets of consecutive edges, arranged according to the directions:

- For $-m \leq j \leq m$, the set of edges in direction j is $S_j = B_{j,|j|}$. (See Notation 4.)
- For $j = m + i$, $0 \leq i \leq 2k$, the set of edges in direction $m + i$ is $S_{m+i} = B_{m+i, m-\epsilon}$, where

$$\epsilon = \begin{cases} 0, & \text{if } 2 \mid i, \\ 1, & \text{if } 2 \nmid i. \end{cases}$$

Note that the sets S_m and S_{m+2k} are defined twice: S_m appears as $B_{m,m}$ in both clauses, while S_{m+2k} appears as $S_{-m} = B_{-m,m}$ in the first clause and as $S_{m+2k} = B_{m+2k,m}$ in the second clause (since $-m$ and $m + 2k$ are the same direction).

As the structure of $G_{n,k}$ is a bit complex, we give an intuitive explanation of the construction before presenting the proof. Our explanation is illustrated by two examples of $G_{n,k}$, presented in Figure 4.

Intuitive explanation of the construction. We consider the directions sequentially, in the order $0, 1, \dots, m + 2k - 1, -m, -m + 1, \dots, -1, 0$. (Note that direction $-m$ is the same as direction $m + 2k$, and thus, the order of directions we consider is logical.) In each direction, we choose k edges, aiming at choosing the “most central” (i.e., longest possible) edges, subject to the restriction that $G_{n,k}$ must avoid A . The endpoints of the edges in each direction j form two sets of k consecutive vertices (see Figure 4). We call the set which is closer to the leftmost

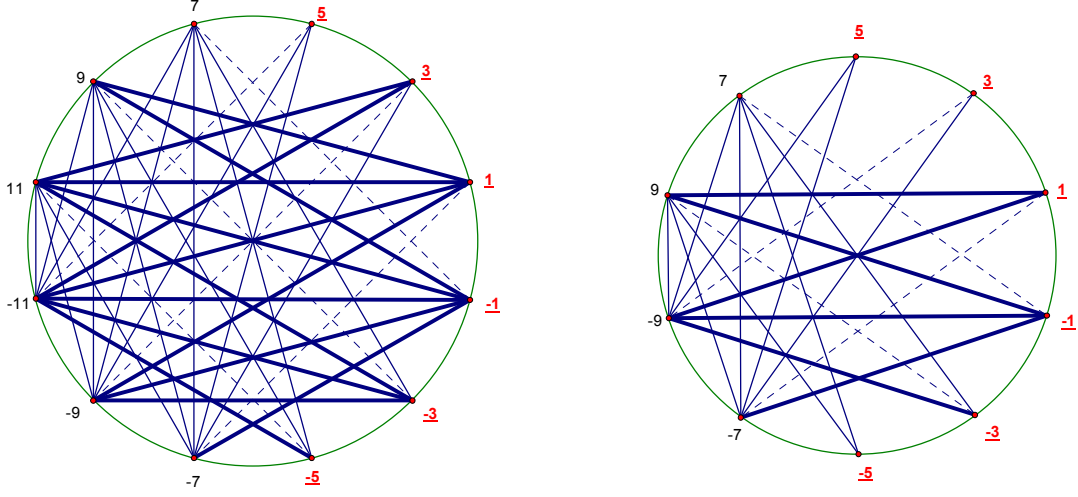


Figure 4: Examples of the CGG $G_{n,k}$. On the left, $n = 12$ and $k = 3$. On the right, $n = 10$ and $k = 2$. In both graphs, the avoided vertices are colored red and underlined. The edges of S_j for $|j| < m$ (where $m = (n - 2k)/2$) are drawn as ordinary lines, the edges of S_j for $m < j < m + 2k$ are drawn as bold lines, and the edges of S_m, S_{m+2k} are drawn as punctured lines.

vertex “the left set” and denote it by L_j , and call the other set “the right set” and denote it by R_j .¹

In the first $m + 1$ directions, the left set $L_j = \{n - 1, n - 3, \dots, n - (2k - 1)\}$ is fixed, while the right set R_j is moved each step one place counterclockwise (i.e., starting with $\{-(n - 1), -(n - 3), \dots, -(n - 2k + 1)\}$, then $\{-(n - 3), -(n - 5), \dots, -(n - 2k - 1)\}$, etc.). We continue in this fashion until $j = m$, for which $S_m = B_{m,m}$ is central (i.e., each of the arcs $Arc_{S_m,m}, Arc'_{S_m,m}$ contains exactly m vertices). In the following directions, we try to maintain centrality: Every second direction is central, and the directions in between are “nearly central”, i.e., moved half a step downward from the central positions. This is achieved by moving L_j and R_j alternately: In passing from S_m to S_{m+1} , L_m is moved one step counterclockwise, while R_m remains unchanged. In passing from S_{m+1} to S_{m+2} , R_{m+1} is moved one step counterclockwise, while L_{m+1} is retained, etc. This continues until we reach direction $m + 2k$ (again central), which is the same as direction $-m$. The transition from here back to direction 0 can be viewed as a mirror image (in reverse) of the transition from S_0 to S_m . In the last step (from S_{-1} back to S_0) R_{-1} is moved one step counterclockwise to R'_{-1} , and then the roles of R and L are interchanged: $L_0 = R'_{-1}$ and $R_0 = L_{-1}$.

3.2 Proof of Theorem 3(1) for Even n

In this section we prove that $G_{n,k}$ satisfies the conditions of Theorem 3(1). It is clear that $G_{n,k}$ contains exactly k edges in each direction² and avoids the set of vertices $A = \{-2m + 1, -2m +$

¹Formally, L_j is the set whose distance to the vertex labelled n is smaller, where the distance $dist(S, v)$ of a set of vertices S from a single vertex v is defined as $dist(S, v) = \min_{s \in S} ord([s, v])$. A single exception is direction 0 for which the distances of both sets from vertex n are equal. For that direction, we choose for convenience $L_0 = \{n - 1, \dots, n - (2k - 1)\}$ and $R_0 = \{-(n - 1), \dots, -(n - 2k + 1)\}$.

²Note that in each direction, there are at least k allowed edges. Indeed, by the definition of A , the set A^c contains $2k$ vertices. Hence, at least k of the edges in each direction have an endpoint in A^c .

$3, \dots, 2m-3, 2m-1\}$. Hence, Theorem 3(1) is implied by the following proposition.

Proposition 8. *For each n and k , the graph $G_{n,k}$ is I_{k+1} -free.*

In the proof of Proposition 8, we use a lemma, which requires some additional notation.

Notation 9. *The endpoints of any edge $e \in E(G_{n,k})$ divide the boundary of P into two open arcs. Denote the arc that does not contain the open boundary edge $]n-1, -(n-1)[$ by Arc_e , and the other arc by Arc'_e . We say that an edge e_2 lies behind e_1 (with respect to $]n-1, -(n-1)[$) if $Arc_{e_2} \subseteq Arc_{e_1}$. (Note that an edge is said to lie behind itself.)*

Note that the notations Arc_e and Arc'_e differ from the notations $Arc_{S_j,j}$ and $Arc'_{S_j,j}$ defined in Section 2.1. This difference is intentional, and both types of notation will be used in the sequel.

Lemma 10. *For any $e \in E(G_{n,k})$, the open arc Arc_e contains at least $m-1$ vertices of G . Furthermore, if e emanates from a positive-labelled vertex, then Arc_e contains at least m vertices of G .*

Proof. We consider separately the sets of edges in S_j for $-m \leq j \leq m$ and for $m+1 \leq j \leq m+2k-1$.

Case I: $e \in S_j$ for $-m \leq j \leq m$. By Definition 7, we have $S_j = B_{j,|j|}$. Recall that, as explained in Section 2.1, $S_j = B_{j,|j|}$ divides the remaining vertices on the boundary of $\text{conv}(V(G))$ into two arcs: $Arc'_{S_j,j}$ (which includes the open edge $]n-1, -(n-1)[$) and $Arc_{S_j,j}$. As by the definition of $B_{j,|j|}$, $Arc'_{S_j,j}$ contains $|j| \leq m$ vertices of G , it follows that $Arc_{S_j,j}$ contains at least $n-2k-|j| \geq m$ vertices of G . Since for each $e \in S_j$, the arc Arc_e includes the arc $Arc_{S_j,j}$, it follows that Arc_e contains at least m vertices of G .

Case II: $e \in S_j$ for $m+1 \leq j \leq m+2k-1$. By Definition 7, $S_j = S_{m+i} = B_{m+i,m-1}$ for odd i and $S_j = S_{m+i} = B_{m+i,m}$ for even i . For $S_{m+i} = B_{m+i,m}$, each of the arcs $Arc_{S_{m+i},m+i}$ and $Arc'_{S_{m+i},m+i}$ contains exactly m vertices of G , and thus, for any $e \in B_{m+i,m}$, the arc Arc_e contains at least m vertices of G . Similarly, for $S_{m+i} = B_{m+i,m-1}$, the arcs $Arc_{S_{m+i},m+i}$ and $Arc'_{S_{m+i},m+i}$ contain $m+1$ and $m-1$ vertices of G respectively, and thus, for any $e \in B_{m+i,m-1}$, the arc Arc_e contains at least $m-1$ vertices of G .

When trying to show that, for some of the edges, the arc Arc_e contains at least m vertices of G , we face a difficulty: Unlike the case $-m \leq j \leq m$, where for any $e \in S_j$ the arc Arc_e includes the arc $Arc_{S_j,j}$, here there is a variance between the edges. For some of the edges e , Arc_e includes $Arc_{S_{m+i},m+i}$ (which contains $m+1$ vertices of G), while for the other edges, Arc_e includes $Arc'_{S_{m+i},m+i}$ (which contains only $m-1$ vertices of G).

However, we observe that the edges e for which Arc_e includes $Arc_{S_{m+i},m+i}$ are exactly those which emanate from positive-labelled vertices (see Figure 5). Indeed, if $e \in S_{m+i}$ emanates from a positive-labelled vertex, then the vertex $m+i$ and the open edge $]-(n-1), n-1[$ lie on opposite sides of the edge e . Thus, by the definitions of Arc_e and of $Arc_{S_{m+i},m+i}$, it follows that $Arc_{S_{m+i},m+i} \subset Arc_e$. On the other hand, if e emanates from a negative-labelled vertex, then $m+i$ and $]-(n-1), n-1[$ lie on the same side of e , and thus, $Arc'_{S_{m+i},m+i}$ is included in Arc_e . Hence, we conclude that for any $e \in E(G_{n,k})$ that emanates from a positive-labelled vertex, Arc_e contains at least m vertices of G . This completes the proof of the lemma. \square

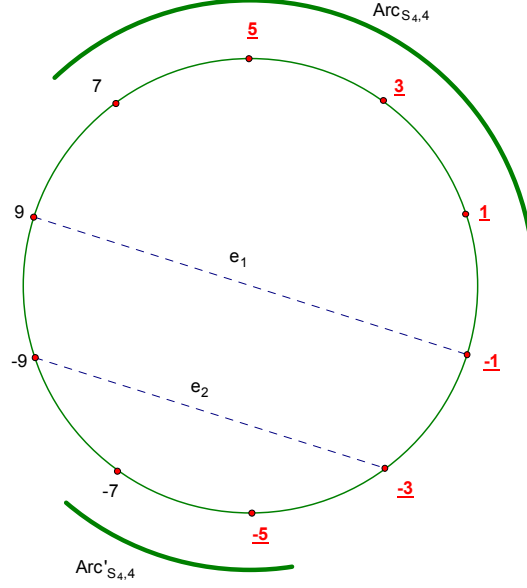


Figure 5: An illustration for the proof of Lemma 10, corresponding to $n = 10$ and $k = 2$. The avoided vertices are colored red and underlined. For the edge $e_1 = [9, -1]$, the arc Arc_{e_1} includes $Arc_{S_{4,4}}$, while for the edge $e_2 = [-9, -3]$, the arc Arc_{e_2} does not include $Arc_{S_{4,4}}$.

Proof. (of Proposition 8)

Assume, on the contrary, that $G_{n,k}$ contains a set S of $k+1$ pairwise disjoint edges. By the construction of $G_{n,k}$, each edge of S has at least one endpoint in the set

$$A^c = \{\pm(n-1), \pm(n-3), \dots, \pm(n-2k+1)\},$$

that consists of k positive-labelled vertices and k negative-labelled vertices (see Figure 6). As the edges of S are disjoint, and there are $k+1$ edges in S , there exist two edges $[i_1, t_1], [i_2, t_2] \in S$, such that:

- $[i_1, t_1]$ does not have a negative-labelled endpoint in A^c . In the notations of Figure 6, $[i_1, t_1]$ does not have an endpoint in D_3 , so it must have an endpoint in D_2 . Moreover, $Arc_{[i_1, t_1]} \subset D_1 \cup D_2$.
- $[i_2, t_2]$ does not have a positive-labelled endpoint in A^c . In the notations of Figure 6, $[i_2, t_2]$ does not have an endpoint in D_2 , so it must have an endpoint in D_3 . Moreover, $Arc_{[i_2, t_2]} \subset D_1 \cup D_3$.

It follows that none of the arcs $Arc_{[i_1, t_1]}, Arc_{[i_2, t_2]}$ is included in the other, and therefore they are disjoint (see Figure 6).

We would like to show that the arc $Arc_{[i_1, t_1]}$ contains at least m vertices of G that are not used by any edge of S , and the arc $Arc_{[i_2, t_2]}$ contains at least $m-1$ such vertices. This will be a contradiction to the assumption $|S| = k+1$, since by the assumption, S uses $2k+2$ vertices of G , and thus, only $n - (2k+2) = 2m-2$ vertices of G are not used by S .

We say that an edge $e \in S$ is *extremal* if no other edge of S lies behind e (see Notation 9). It is immediate that behind any edge of S there is an extremal one. Note that since the edges

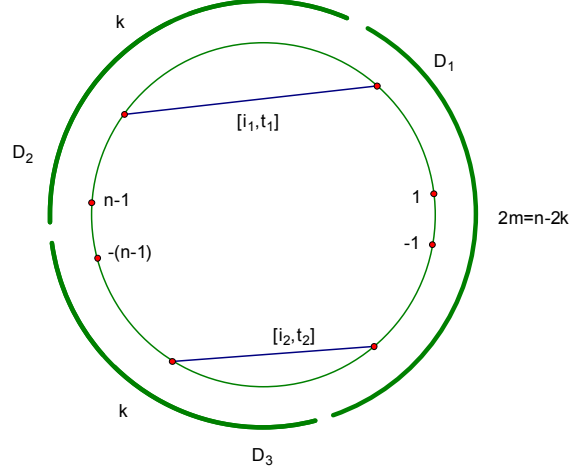


Figure 6: An illustration for the proof of Proposition 8. The avoided set A consists of the arc D_1 (containing $2m = n - 2k$ vertices). The set A^c consists of two arcs: The arc D_2 of k positive-labelled vertices, and the arc D_3 of k negative-labelled vertices. The edges $[i_1, t_1], [i_2, t_2]$ are chosen such that $[i_1, t_1]$ does not have endpoints in D_3 , and $[i_2, t_2]$ does not have endpoints in D_2 .

of S are pairwise disjoint, if e' is an extremal edge of S then the vertices of $Arc_{e'}$ are not used by any edge of S .

Let e_1 and e_2 be extremal edges that lie behind $[i_1, t_1]$ and $[i_2, t_2]$, respectively. (Note that we may have $e_1 = [i_1, t_1]$ and/or $e_2 = [i_2, t_2]$.) By Lemma 10, each of the arcs Arc_{e_1} and Arc_{e_2} contains at least $m - 1$ vertices of G , and these vertices are certainly not used by any edge of S . Moreover, since both endpoints of e_1 are contained in $Arc_{[i_1, t_1]} \subset \{n - 1, n - 3, \dots, 1, -1, \dots, -(n - 2k - 3), -(n - 2k - 1)\}$, it follows that e_1 emanates from a positive-labelled vertex. (This holds since e_1 must emanate from a vertex in A^c , and $Arc_{[i_1, t_1]} \cap A^c$ contains only positive-labelled vertices.) Hence, Lemma 10 actually implies that Arc_{e_1} contains at least m vertices of G that cannot be used by edges of S . In total, we have at least $m + (m - 1) = 2m - 1$ vertices of G that cannot be used by edges of S (and these vertices are distinct since $Arc_{e_1} \cap Arc_{e_2} \subset Arc_{[i_1, t_1]} \cap Arc_{[i_2, t_2]} = \emptyset$). This leads to a contradiction as explained above, and thus concludes the proof of Proposition 8. \square

3.3 Maximal CGGs Avoiding $n - 2k$ Consecutive Vertices for Odd n

The case of odd n is very similar to the case of even n , and is, in some respect, even easier. Thus, we only sketch briefly the required modifications in the notations, in the construction of $G_{n,k}$, and in the proof of Theorem 3(1).

Notations. The first slight change is in the notations. Let n be odd, and let $m = \lfloor (n - 2k)/2 \rfloor = (n - 2k - 1)/2$. As $|A| = 2m + 1$ is odd, we take the vertices of $V(G)$ to be the even-labelled vertices of P , as shown in Figure 7. This is in contrast with the case of even n , where $|A|$ is even, and the vertices of $V(G)$ are taken to be the odd-labelled vertices of P (see Section 2.1).

Construction. The second change is in the definition of $G_{n,k}$. While the definition is very similar to the even case, there are a few changes. The formal definition of $G_{n,k}$ in the odd case

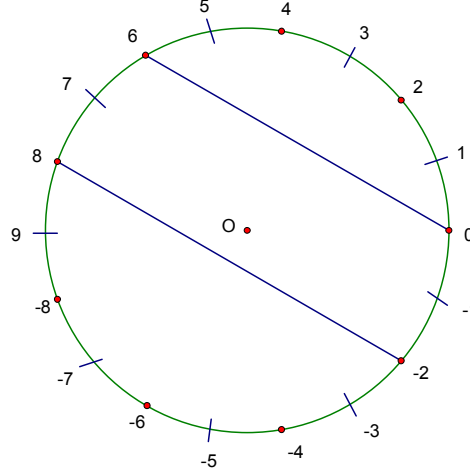


Figure 7: Representation of a CGG G on 9 vertices using a regular polygon P on 18 vertices. The vertices of G are the even-labelled vertices of P . The set of edges shown is $B_{3,3}$.

is as follows:

Definition 11. For an odd integer n , and $k \leq \lfloor \frac{n}{2} \rfloor - 1$, we denote by $G_{n,k}$ the CGG on n vertices whose edge set is the union of the following sets of consecutive edges, arranged according to the directions:

- For $-(m+1) \leq j \leq m+1$, the set of edges in direction j is $S_j = B_{j,|j|}$.
- For $j = m+i$, $0 \leq i \leq 2k+1$, the set of edges in direction $m+i$ is $S_{m+i} = B_{m+i, m+\epsilon}$, where

$$\epsilon = \begin{cases} 0, & 2 \mid i, \\ 1, & 2 \nmid i. \end{cases}$$

(Note that the edges in directions $\pm m, \pm(m+1)$ are defined twice. Of course, the definitions coincide.) An example of $G_{n,k}$, with $n = 13$ and $k = m = 3$, is presented in Figure 8.

Main Lemma. The main lemma used in the proof of Theorem 3(1) in the even case, Lemma 10, is replaced by the following lemma, which turns out to be even stronger:

Lemma 12. For any $e \in E(G_{n,k})$, the open arc Arc_e contains at least m vertices of G .

To prove Lemma 12, we note that in the odd case, a set of the form $B_{j,m}$ or $B_{j,m+1}$ leaves m and $m+1$ vertices of G on its sides (compared to m and m or $m-1$ and $m+1$ in the even case). As a result, since for any j such that $m \leq j \leq m+2k+1$, S_j is of one of the forms $B_{j,m}$ or $B_{j,m+1}$, it follows immediately that for any edge $e \in S_j$ (for $m \leq j \leq m+2k+1$), Arc_e contains at least m vertices of G . The proof for $-m < j < m$ is identical to the proof of the corresponding statement of Lemma 10 in the even case.

Proof of Theorem 3(2). Due to the stronger form of Lemma 12, the proof is even simpler than in the even case. Same as in the even case, we assume on the contrary that $G_{n,k}$ contains a set S of $k+1$ pairwise disjoint edges, and show that S contains two edges $[i_1, t_1], [i_2, t_2]$ such that $[i_1, t_1]$ does not have a negative-labelled endpoint in A^c , and $[i_2, t_2]$ does not have a positive-labelled endpoint in A^c . Furthermore, each of the closed arcs $Arc_{[i_1, t_1]}, Arc_{[i_2, t_2]}$ contains the

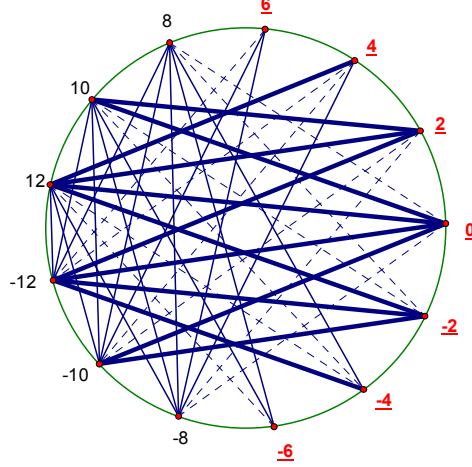


Figure 8: An example of the CGG $G_{n,k}$, for $n = 13$ and $k = 3$. The avoided vertices are colored red and underlined. The sets S_j for $|j| < m$ are drawn as ordinary lines, the sets S_j for $m+1 < j < m+2k$ are drawn as bold lines, and the sets $S_m, S_{m+1}, S_{-m}, S_{-m-1}$ are drawn as punctured lines.

endpoints of an extremal edge (which may be one of the edges $[i_1, t_1], [i_2, t_2]$). We denote these extremal edges by e_1 and e_2 , respectively. By Lemma 12, each of the arcs Arc_{e_1}, Arc_{e_2} contains at least m vertices of G that are not used by edges of S . Hence, the number of vertices that are used by edges of S is at most $n - 2m = 2k + 1$, contradicting the assumption that S consists of $k+1$ disjoint edges. This completes the proof of the theorem for odd n .

4 CGGs Avoiding $n - 2k + \ell$ Consecutive Vertices

In this section we consider I_{k+1} -free CGGs on n vertices that admit a free boundary arc of order $n - 2k + \ell$, for $1 \leq \ell < k$. As shown in Section 2, the number of edges in such a graph is at most $kn - \ell(\ell + 1)/2$. We show that this upper bound is attained by a sequence of graphs $G_{n,k,\ell}$, thus proving Theorem 3(2).

Same as in Section 3, we write $m = \lfloor (n - 2k)/2 \rfloor (= \lfloor n/2 \rfloor - k)$, so that $|A| = 2m + \ell$ if n is even, and $|A| = 2m + \ell + 1$ if n is odd. As indicated in Section 2.1, we take $V(G)$ to be the set of odd-labelled vertices of P if $|A|$ is even, or the set of the even-labelled vertices of P if $|A|$ is odd. We define $K = A^c$, and divide K into three blocks: K_+ (the uppermost $k - \ell$ vertices of K), K_- (the lowest $k - \ell$ vertices of K), and K_0 (the ℓ vertices in the middle), see Figure 9.

For ease of exposition, we consider the cases of even and odd n separately. In Sections 4.1 and 4.2 we present the construction of $G_{n,k,\ell}$ for even values of n and prove that these CGGs satisfy the conditions of Theorem 3(2). In Section 4.3 we sketch the modifications in the construction of $G_{n,k,\ell}$ and in the proof of the theorem required in the case of odd n .

4.1 Construction of the Graphs $G_{n,k,\ell}$ for Even n

For $n = 2k + 2m$ even, we describe the edges of $G_{n,k,\ell}$ by direction, as follows. The directions range over the half-circle from $-\ell - m$ up to $2k + m - \ell (= -\ell - m + n)$. The set of edges of $G_{n,k,\ell}$ in direction j is denoted by S_j . We divide the directions into five sub-ranges:

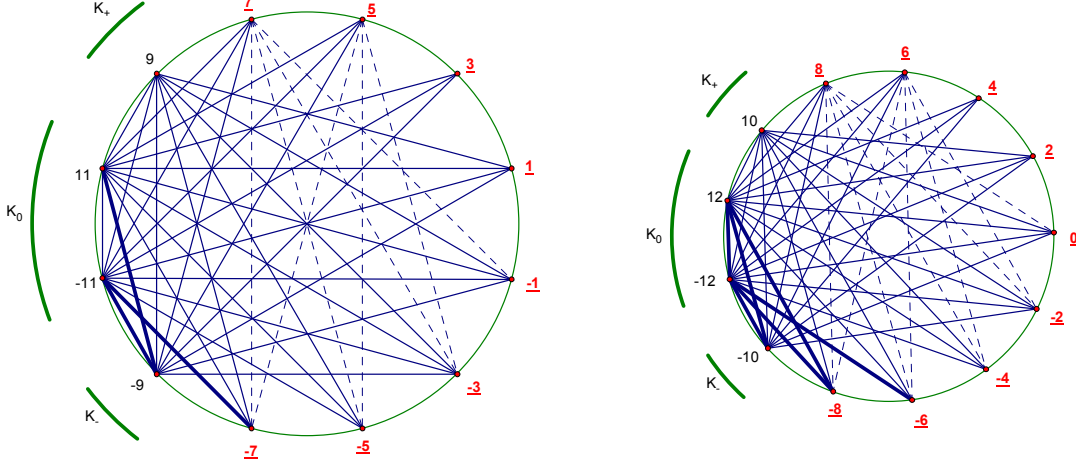


Figure 9: The CGG $G_{n,k,\ell}$ for different values of n, k , and ℓ . On the left: $n = 12, k = 3$, and $\ell = 2$. On the right: $n = 13, k = 3$, and $\ell = 2$. In both figures, the avoided vertices are colored red and underlined. The edges of $G_{n,k,\ell}$ that do not belong to $G_{n,k}$ are denoted by bold lines. The edges of $G_{n,k,\ell}$ that belong to $G_{n,k}$ are denoted by ordinary lines. The edges of $G_{n,k}$ that do not appear in $G_{n,k,\ell}$ are denoted by punctured lines. The sets K_+, K_- , and K_0 are marked in both graphs.

1. $-\ell - m \leq j \leq -\ell$,
2. $-\ell \leq j \leq 0$,
3. $0 \leq j \leq \ell$,
4. $\ell \leq j \leq \ell + m$,
5. $\ell + m \leq j \leq 2k + m - \ell$.

The edges in the subranges are defined as follows:

- $j = 0$: For $j = 0$, S_0 consists of the $k - \lceil \ell/2 \rceil$ leftmost vertical edges.
- $0 < j \leq \ell$: We enlarge K by adding j vertices at the bottom. S_j consists of the $k - \lceil (\ell - j)/2 \rceil$ parallel edges that fit into this enlarged arc, with one free vertex in the middle if $\ell - j$ is odd.
- $-\ell \leq j < 0$: Same as above, with “bottom” replaced by “top” and j replaced by $|j|$.
- $\ell \leq j \leq \ell + m$: We enlarge K by adding j vertices at the bottom. S_j consists of the k most central parallel edges that fit into this enlarged arc, leaving $j - \ell$ free vertices in its center. ($S_{\ell+m}$ is already central.)
- $-\ell - m \leq j \leq -\ell$: Same as above, with “bottom” replaced by “top”, and j by $|j|$.
- $\ell + m \leq j < 2k + m - \ell$: S_j is a k -block, central if $j \equiv \ell + m \pmod{2}$, and near central if $j \equiv \ell + m + 1 \pmod{2}$. In the latter case, there are $m + 1$ unused vertices of G above S_j , and $m - 1$ below.

The idea behind the construction is the following: For each direction j , we consider the set of allowed edges (i.e., edges with at least one endpoint included in A^c). The edges of this set are consecutive. If their number is at most k , we take all of them to $G_{n,k,\ell}$. If their number exceeds k , we take the k edges which are the closest to the center, and if there are two k -blocks with the same distance from the center, we take the lower one among them.

Another way to view the construction is to compare $G_{n,k,\ell}$ to the corresponding graph $G_{n,k}$. When moving from $G_{n,k}$ to $G_{n,k,\ell}$, ℓ vertices are added to the forbidden set A . As a result, we lose part of the edges that emanate from these ℓ vertices. When defining $G_{n,k,\ell}$, we retain all edges of $G_{n,k}$ that remain allowed, and add some edges to compensate for the edges that become forbidden. Specifically, in each direction j in which all edges of $G_{n,k}$ remain allowed, we take S_j to be the set of edges of $G_{n,k}$ in that direction. In the directions in which some of the edges of $G_{n,k}$ become forbidden, we observe that these edges are those emanating from a set of consecutive vertices. For each such direction, we remove from the edges of $G_{n,k}$ the edges that become forbidden and compensate for them by adding a set of consecutive edges on the other side of the set of $G_{n,k}$. If the number of edges we can add is smaller than the number of edges that become forbidden (since the direction “ends”, in a single vertex or in a boundary edge), we add the maximal possible number of edges.

As a result, $G_{n,k,\ell}$ satisfies the following, for each j :

- The set of edges of $G_{n,k,\ell}$ in direction j is consecutive.
- If at most k edges in direction j are allowed, then $G_{n,k,\ell}$ contains all allowed edges in direction j . Otherwise, $G_{n,k,\ell}$ contains k edges in direction j .

An example of the construction $G_{n,k,\ell}$, for $n = 12$, $k = 3$, and $\ell = 2$, is presented in the left part of Figure 9. In the figure, in directions $-5, -4, -3, -2, 4, 5, 6$ all edges remain allowed, in directions $-1, 2, 3$ a single edge becomes forbidden, and in directions $0, 1$ two edges become forbidden. On the other hand, a single compensating edge is added to each of directions $0, 1, 2, 3$. As a result, $G_{12,3,2}$ contains 2 edges in directions $-1, 0, 1$, and 3 edges in each other direction.

4.2 Proof of Theorem 3(2)

In this section we present the proof of Theorem 3(2) in the case of even n .

Assume $n = 2m + 2k$, $m, k \geq 2$, and $1 \leq \ell < k$. We have to establish three claims about the graph $G_{n,k,\ell}$ constructed above:

1. $G_{n,k,\ell}$ is I_{k+1} -free.
2. $G_{n,k,\ell}$ has $nk - \binom{\ell+1}{2}$ edges.
3. $G_{n,k,\ell}$ has a free boundary arc of order $2m + \ell$.

Item (3) is obvious, since, by our construction, A is a free arc of $G_{n,k,\ell}$.

Item (2): The number of edges of $G_{n,k,\ell}$ in direction j is: $k - \lceil \ell/2 \rceil$ for $j = 0$, $k - \lceil (\ell - |j|)/2 \rceil$ for $1 \leq |j| \leq \ell$, and k for all other $n - (2\ell - 1)$ directions. It follows that

$$e(G_{n,k,\ell}) = nk - 2 \sum_{i=1}^{\ell-1} \lceil i/2 \rceil - \lceil \ell/2 \rceil = nk - \ell(\ell + 1)/2,$$

by Claim 6.

Item (1): This is the main claim. Assume, on the contrary, that S is a set of $k+1$ pairwise disjoint edges of $G_{n,k,\ell}$. Since $|K_0 \cup K_-| = k$, there is an edge $e^+ \in S$ that does not use any vertex of $K_0 \cup K_-$. It follows that e^+ is not vertical, that the left vertex of e^+ belongs to K_+ , and that its right vertex belongs to K_- or to A . Among all edges in S whose left endpoint is in K_+ , choose the one whose left endpoint is as far to the right as possible and call it e_+ . By the construction of $G_{n,k,\ell}$, there are at least m vertices above e_+ that are not used by any edge in S .

Similarly, there is an edge $e^- \in S$ that does not use any vertex in $K_0 \cup K_+$. Choose such an edge e_- whose left endpoint is as far to the right as possible. By the construction of $G_{n,k,\ell}$, there are at least $m-1$ vertices below e_- that are not used by any edge of S . It follows that S uses altogether at most $n - m - (m-1) = 2m + 2k - (2m-1) = 2k+1$ vertices, a contradiction.

4.3 Maximal CGGs Avoiding $n - 2k + \ell$ Consecutive Vertices for Odd n

The construction of $G_{n,k,\ell}$ in the odd case is almost exactly the same as in the even case. The only difference in the definition is in the last sub-range of directions, namely, $\ell + m \leq j < 2k + m - \ell$. In the odd case, the directions $2k + m - \ell, 2k + m - \ell + 1$ are added to that sub-range, and (unlike the even case), S_j is always an almost central k -block, such that there are m unused vertices of G above it if $j \not\equiv \ell + m \pmod{2}$, and $m+1$ unused vertices above it if $j \equiv \ell + m \pmod{2}$.

An example of $G_{n,k,\ell}$, with $n = 13, k = 3$, and $\ell = 2$, is presented in the right part of Figure 9. In the figure, in directions $-5, -4, -3, -2, 5, 6, 7$ all edges remain allowed, in directions $-1, 3, 4$ a single edge becomes forbidden, and in directions $0, 1, 2$ two edges become forbidden. On the other hand, a single compensating edge is added to each of directions $0, 1, 3, 4$, and two compensating edges are added to direction 2 . As a result, $G_{13,3,2}$ contains 2 edges in directions $-1, 0, 1$, and 3 edges in each other direction.

The proof of Theorem 3(2) in the odd case is almost identical to the proof in the even case. The only difference is that in the odd case, both e_+ and e_- leave at least m unused vertices of G behind them. This implies that at most $n - 2m = 2k + 1$ vertices of G are used by edges of B , a contradiction to the assumption that the $k+1$ edges in B are disjoint. This completes the proof of Theorem 3(2) for odd n .

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